A Proof of the Theorem

THE EQUATION $f(z) = 0$ HAS A ROOT WHERE $f(z)$ IS ANY HOLOMORPHIC FUNCTION OF z.

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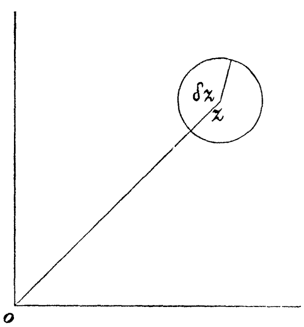
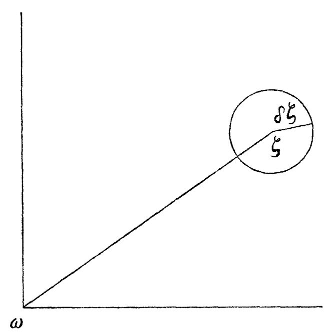
Represent z by a point in one plane and $f(z) = \zeta$ by a point in another plane.

If $f(z) = \zeta$ cannot become equal to zero for any value of z, there must be some minimum distance from the origin ω within which ζ cannot fall. Suppose ζ at such minimum distance from ω.

We can suppose this, for, $f(z)$ becoming infinite along with z, this minimum must be for a finite value of z, and can therefore be reached.





To z give an increment δz; the corresponding increment of ζ is $\delta\zeta = \frac{f^r(z)}{\underline{|r}}(\delta z)^r$ where $f^r(z)$ is the first of the successive derivatives of $f(z)$ which does not vanish for this value of z. Now varying δz, make it describe a closed curve round z; $\delta\zeta$ will at the same time describe a closed curve r times over round the point ζ, and will therefore come between the point ζ and origin ω. The point ζ has then no such minimum distance from the origin ω as was supposed; the function $f(z) = \zeta$ is therefore capable of becoming $= 0$, and the equation $f(z) = 0$ has therefore a root.

The above statement might be slightly varied, thus: to $f(z)$ we can give an increment in any direction we may choose; for δz, $\delta\zeta$, being any two corresponding increments of z, ζ, respectively; if the required increment is to be in a direction inclined to $\delta\zeta$ at an angle α, give to δz the rotation $\frac{\alpha}{r}$ and $\delta\zeta$ takes the required direction.

If mod (δz) remain constant while rotating round z, $\delta\zeta$ at the same time describes a circle round ζ, and we give ζ as much of an increment in one direction as in another.

We can always for a polynomial take $r = 1$, for if there be no root of $f(z) = 0$, ζ will have a minimum along each of radiating lines drawn through origin ω (for $f(z)$ can easily be shown to have some value on each of these lines), and as $f'(z)$ cannot vanish for more than $n - 1$ values of z, we can always choose our minimum along a line on which $f'(z)$ cannot vanish.

[Just as this note was about to go to press, I discovered that practically the same proof as above had been given by Hoüel in his Cours de Calcul Infinitésimal.—J. C. F.]